

Origami Modeling of Functional Structures based on Organic Patterns

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Shape-changeable origami models and rigid core models are analytically designed by folding or creasing a flat sheet or a thin plate into a 3-dimensional shape. The former models are designed by analyzing and mimicking the helical patterns or structures often found in living organisms, because such helical structures are easily deployed due to the presence of fewer dynamic restraints. Models presented here include foldable/deployable thin cylinders, conical-shaped membranes, circular sheets which may be folded or wrapped, foldable 3-D structures such as elliptical or spherical shell, and a twisted tube and cone. Some of these models are used to interpret the mechanics of the unfolding of flower buds and insect wings. Ultra-lightweight 3-D core models mimicking a beehive's honeycomb structure and sponge type core are also presented for use in building aerospace structures.

1. Introduction

In an attempt to establish future low-cost deployable aerospace structures, inflatable techniques for erecting these structures were tested by Spartan 207 satellite in 1996¹. In the future, launchings of lightsails consisting of a huge membrane are planned. To stow such large structures efficiently in rockets, the development of folding/deploying techniques and ultra-lightweight structures are of prime importance. Analytical origami modeling may provide a powerful tool to satisfy these demands.

In the phyllotaxis of leaves or flowers, and in some flower buds and insect wings, helical patterns often appear. It is well known that DNA and proteins such as collagen consist of helical structures which incorporate ingenious folding mechanism. These helical-type structures may relate to *change* such as deformation, growth or evolution. By analyzing these helical patterns, easily deployable structures can be developed. However, this type of analytical work is as yet in its infancy²⁾, and very few analytical models have been reported^{3~7)}.

This research aims to develop the functional ability of origami structures with foldability/deployability and to devise rigid, light cores by mimicking and improving upon organic patterns. We develop models for foldable tubes, conical shells, circular membranes, movable/shape-changeable models and highly rigid 3-D cores.

2. Basic Patterns and folding conditions

Helical patterns are typically seen in the cactus, the pineapple, the pinecone, the sunflower and in cross-sections of the shell of the nautilus (Fig.1 (a)~(e)). It is well known that the numbers of helices in the pineapple, the pinecone and the sunflower patterns consist of 2 adjoining numbers in the Fibonacci series (f_i ; 1,2,3,5,8,13,21,34,...). The ratio of these 2 numbers (f_{i+1}/f_i) becomes golden number $\tau = (\sqrt{5}+1)/2 \approx 1.618$, when $i \rightarrow \infty$; plant geometry has often been explained by the golden number or the golden angle^{8~10}. Consider a right-angled triangle ABC (Fig.1 (f)) with the side lengths AC=13 and BC=21. By drawing 21 and 13 equidistant parallel lines right and left upwards, respectively, draw a parallelogram ABDC. We denote the intersection points of these lines primordia. Gluing segment AC to BD of Fig. 1(f) gives a tube with 21 and 13 helices. It may be seen that the slope-angle and the pattern of the cactus pictured here correspond to Fig.1(f). Consider now introducing diagonal lines through the squares (dotted lines) as shown in Fig.1 (f). Fundamental phyllotactic patterns as well as origami models consist of these 3 kinds of helices. These patterns are characterized by the fact that the distal ends of the helices appear at free boundary of top and bottom edges of the tube. A plastically-buckled PVC tube and a Cu-made conical shell by axial compression are shown in Figs. 1(h) and (j). Fig.1 (i) and (k) show re-stretched or cut and opened appearances of buckled samples. Fig.1(i) shows that a tube typically buckles like an origami but with large plastic deformation near fold lines. Note that circumferential (horizontal) valley creases close by themselves; this kind of pattern is called *symmetrical* in this article. Origami models of this type lack in deployability, while helical models such as in Fig.1(f) deploy very well because helices do not have dynamical restraints at their distal ends(see Fig.1(f) right).

Setting AC/BC=8/13, construct parallel lines between both pairs of sides (Fig.1(g)). By connecting the points of intersection (named node in origami models), the chart can be tessellated by such patterns as ① triangles formed by diagonals of parallelograms, ② equilateral trapezoids, ③ twisted trapezoids (bisected Dirichlet domains) or ④ combinations of 2 kinds of triangles. After considering buckling patterns, this article will proceed to consider the more completed helical studies. Before designing the models, the folding conditions at the nodes must be explained. When a flat sheet is folded perfectly (flat folding) at a node, the numbers of creases n is even number. The conditions for $n=4$ and 6 are given by $\alpha = \gamma$ (or $\gamma + \delta = \pi$) and $\beta - \alpha = \delta - \gamma + \theta$, respectively, using the angles denoted in Fig.2 (a) and (b) (solid line; mountain (M) fold, dotted line; valley (V) fold⁵). These relations can be derived using the flat folding condition that the sum of the alternate angles around a node (supplementary angle) is π . When the folding condition in Fig.2 (a) holds, another folding as in Fig.2(c) is possible. We call this *dual foldability*, which is different from *reversibility* in reference 2. Fig.2 (d) shows that when 4 fold lines are drawn by reflecting on 2 orthogonal lines of reflection, flat folding is automatically possible (named orthogonal mirror law). In the case of 3-D revolution (Fig. 2(e)), cut it by the planes orthogonal to Z-axis, and successively divide the obtained fan-shaped strips into small elements of rough parallelograms as ①, ②...;

replace the revolution by polyhedron. Because the original surface is curved, there are wedge-shaped space between four small elements placed on a plane as in Fig.2 (f). The folding condition for this case is $\alpha + \gamma = \beta + \delta$ using the angles denoted in the figure.

3. Design of foldable/deployable structures

3.1 Foldable cylinders and conical-shaped membranes

With reference to Fig.1 (i), a symmetrical-type foldable tube is designed. Fig. 3(a) shows an example consisting of 3 triangles in the horizontal direction. Inclined M-creases make angles α and β with horizontal V-creases. The folding conditions at the nodes hold because $\theta = 0$ in the condition given in Fig.2(b). Cutting the structure of Fig.3 (a) at the segment AB and CD gives Fig.3(b). Furthermore by cutting this pattern at BE or BF, another type of helical model is derived. Fig.3(c) is the pattern obtained by cutting the pattern of Fig.3 (b) at BF, where the base helix ① spirals up 2 steps by each turn ($S=2$). By taking the bottom row of Fig.3(b) and folding it, the right and the left sides make an angle 12α . To make this axially foldable, the condition $12\alpha = 360^\circ$ must hold adding the folding condition at all nodes. This is called *the circumferential closing (CC) condition*⁵⁾. The angle α is generally given by $360^\circ / 2N$ (N ; the number of elements in the horizontal direction, β ; arbitrary). For the symmetrical models of Fig.3(a), the CC condition is $\alpha + \beta = 60^\circ$. By numerical calculations, Guest and Pellegrino obtained 3 sets of (α, β) values for the folding of the Fig.3(c)-type helical model. By origami theory, the CC condition is easily derived as $N\alpha - S\beta = 180^\circ$ (S ; spiral-up number). Note that the CC condition is independent for each row; inverting type helical patterns such as Fig.3 (d) are also foldable. Fig.3(e) gives a symmetric type foldable square tube ($\alpha = 45^\circ$). By cutting this chart at a steep line while keeping continuity of the creases at both sides, it becomes helical type (Fig.3 (f)). A foldable tube with the trapezoidal pattern of Type ② in Fig.1 (g) is shown in Fig.3(g) ($N=8, \alpha = 22.5^\circ$). Fig.3(h) shows a triangular tube ($N=3, \alpha = 60^\circ$). Foldable tubes of different patterns corresponding to Types ③ and ④ of Fig.1 (g) are shown in Figs.3(i) and (j). Figs.3 (h) and (j) are designed to imitate the triple helices of collagen and double helices of DNA, respectively.

The fundamental chart of a foldable conical shell corresponding to Fig.3 (a) is shown in Fig 3(k). The CC condition of the model is $\alpha + \beta = (360^\circ - \Theta)/N$ (Θ ; the vertical angle of the diagram). Fig.3 (l) and (m) correspond to Fig. 3(b) and (d), respectively. The CC condition for these helical models is $\alpha = (360^\circ - \Theta)/(2N)$ (β ; arbitrary). Figure 3 (n) corresponds to Fig 3(i). Charts for the spiral up model with $S=1$ designed using triangle and trapezoidal elements are shown in Fig 3(p) and (q), which are obtained by numerical calculations. Fig 3(r) is designed to give the buckling model of Fig.1(k). Some conical patterns can be directly related to tubular ones through complex-logarithmic mathematical transformations¹¹⁾.

3.2 Folding/wrapping of a circular sheet

Fig.4 (a) shows an imitative pattern of sunflower (Fig.1 (d)) consisting of 21 and 34 helices in clockwise and anticlockwise directions, respectively. By analyzing such a pattern, folding /wrapping models will be developed (Fig.4 (a)). As in Fig.4(b), draw a segment AB at an angle to the radius ($\angle AOB=n\Theta$) from point A on the periphery of a circle (O; the center). From point B, draw another segment BC with the same angles ϕ and $n\Theta$. By continuing this procedure, an equiangular spiral type (main) fold line ① is defined. Assign another point E on the periphery. Draw a segment EF at an angle ϕ to the radius ($\angle EOF=m\Theta$) and draw a segment FG at an angle χ to the radius ($\angle FOG=m\Theta$). By the same procedure taking ϕ and χ alternately, points G, H · · are defined (named sub-fold line ②). When fold line ② reaches the points B,C or D · · , after zigzagging $M/2$ times, the two helical fold lines coincide at a node. We denote helical models of the fold lines ② reaching the points B,C or D...by 1,2 or 3 step up model ($S=1,2,3\dots$). From points H, J, draw fold lines ③,④ like ①, and the helical mesh pattern is completed. Using $\angle ABH=\phi+\phi+(m+n)\Theta$ and $\angle JBC=\phi+\phi$, the folding condition at a representative node B is given by the following equation (see Fig.2(a)).

$$\phi+2\phi+\chi+(m+n)\Theta=180^\circ \quad (1)$$

Setting the radius of the circle R_0 , the radii of points B, F and G as R_1, R_2, R_3 , respectively, and applying the sine formula to $\triangle OAB$, $\triangle OEF$, and $\triangle OFG$, $r=R_1/R_0$, $p=R_2/R_0$, $q=R_3/R_2$ are expressed by

$$r=\sin\phi/\sin(\phi+n\Theta), \quad p=\sin\phi/\sin(\phi+m\Theta), \quad q=\sin\chi/\sin(\chi+m\Theta). \quad (2)$$

The radii of point trains E, F, G, H ...on ② are therefore given by 1, p, pq, p^2q , p^2q^2 , · · . When ② reaches an S step-up point on ①, the radius of the point is r^S . We therefore have

$$r^S=(pq)^{M/2} \quad (3)$$

Upon setting the number N of main fold-lines (① and ⑤); the circular diagram is divided into N large elements. We further divide them into M small elements (zigzagging number), then the central angle of large element is $(Mm+Sn)\Theta$ for S step-up model; the following relation holds for angle distribution on the circular diagram.

$$N(Mm+Sn)\Theta=360^\circ \quad (4)$$

By selecting $S, M, N, n\Theta$ and $m\Theta$ and setting one of the angle ϕ , ϕ or χ , the above 4 equations can be solved by straightforward numerical calculations. The model of Fig.4(a) was drawn as follows ($N=1$); set the divergence angle $(13m+8n)=360^\circ/\tau^2=137.507^\circ$ (golden angle). This relation together with Eq.(4) with $M=34$ and $S=21$, give $m\Theta=7.66305^\circ$ and $n\Theta=4.73602^\circ$ (note that obtained m/n is τ) The golden number found here appears for any Fibonacci combination. Substituting the m, n, S and M into Eqs.(1) and (3), and applying Eq.(2) with $\phi=\chi$, we drive the values of ϕ and p ($\phi \doteq 29.341^\circ$, $p=0.9205$). Radii r of all the node are different ($r=p^{t/21}$, $t=0,1,2,3$) and they appear rotating around the center by the golden angle (see the path A-C-D in Fig 4(a), AC; $13m\Theta$, CD: $8n\Theta$). It has been shown that when sub-fold lines are equiangular ($\phi=\chi$) as

this model, a wrapped shape also becomes equiangular and its size is $R_0 \sin \phi^{7)}$; the angle ϕ decides wrapping size. By using a smaller $n\Theta (=1^\circ)$ giving $\phi \doteq 13.135^\circ$, wrapping efficiency is improved. The corresponding diagram together with wrapped sample photo is shown in Fig.4 (c). By using formulated equations, various kinds of folding are possible. Firstly, 4 typical models consisting of equiangular sub-fold lines ($\phi = \chi$) are shown. Putting $N=12, M=1, S=1, (m+n)\Theta=30^\circ$, $\phi = 10^\circ$, Fig.4(d) is derived. Fig.4(e) is for $N=2, M=11, S=1, m\Theta=15^\circ, n\Theta=15^\circ$, while Fig 4 (f) is for $(N, M, S)=(3,6,1)$ with $(m\Theta, n\Theta) = (18^\circ, 12^\circ)$ and Fig 4(g) is for $(4,6,1)$ and $(14^\circ, 6^\circ)$. Figs.4(e)~(g) are folded in a plane ($N=2$), wrapped in triangular ($N=3$) or squarely ($N=4$) as shown in their respective photos. By using $(N,M,S)=(2, 6, 1)$ and $(m\Theta, n\Theta) = (29.5^\circ, 3^\circ)$ with $\phi=5^\circ$. Fig.4(h) is obtained. It is folded in a plane but within a smaller size than that in Fig.4(e). Fig.4(i) is obtained for parameter values $(9,2,1)$ and $(18^\circ, 4^\circ)$ with $p=1$. It is wrapped in a regular nonagon sliding downwards. The sliding results from $\angle AOB > \angle BOC$ which arises because $\phi \neq \chi$, while the models with $\phi = \chi$ as in Figs.4(e)~(g) are wrapped symmetrically in the vertical direction. Furthermore they are wrapped in equiangular spiral shapes; wrappings are very tight in the central region when the sheet is not thin enough. To improve this, equiangular main fold lines are modified as below. By taking $\phi = \phi_0 = 10^\circ$ on the periphery (point A in Fig. 4(j)) at first and then by increasing the angles as $11, 12^\circ, \dots$ with every node (points 1,2,...), satisfying the folding condition at these nodes, a modified helical pattern is designed (Fig.4(j)) where main fold lines as ① take a detour in order to reach the center (compare with ① in Fig.4(d)), and a circular sheet is wrapped in nearly equidistant clearances. In these wrapping models, the size of folding angles at sub-fold lines are small so that they can replace by elastic deformation of membranes; by creasing only the main fold lines, they can be effectively wrapped for practical applications. Based on this concept, taking membrane thickness into account, a design program using numerical calculations has been developed to wrap closely circular sheets¹²⁾.

3.2 Wrapping models by Archimedean spiral configurations

Circular sheets are also wrapped by Archimedean-type spiral configurations⁶⁾. As shown in Fig. 5(a), draw pairs of M and V fold lines shown by ① and ② (segments AD and AE, BF and BG) from corners of N -gon shaped central-hub ($\angle BAE = \angle CBG = \alpha$, $\angle EAD = \angle GBF = \beta$). Start a drawing of another fold line (segment BH) from a corner B with arbitrary angle ($\gamma = \angle BHA$) to the side AB of the hub, and extend the fold line symmetrically to the fold lines ① and ② ($\angle BHA = \angle AHI$, $\angle AIH = \angle AIJ$). This fold line named ③ becomes Archimedean type spiral. The M/V versions of ③ are interchanged whenever it intersects lines ① and ②. By folding the chart of Fig.5(a), the circular sheet is wrapped around central hub. The condition to wrap a sheet around central hub symmetrically in a vertical axis of a hub is given by

$$\beta + \gamma = (\pi / 2)(1 + 2/N). \quad (5)$$

By using this relation, we can design various kinds of wrapping models freely by choosing angles

β and γ . Representing examples are shown in Fig.5(b)~(i). Fig.5(b) is depicted putting $N=6$ and $\gamma=90^\circ$ ($\beta=30^\circ$) by taking arbitrary chosen angle α as 15° , and fold lines ③ show Archimedean spiral. Fig.5(c) and (d) are for $(N, \beta, \gamma, \alpha)=(4, 40^\circ, 95^\circ, 30^\circ)$ and $(6, 20^\circ, 100^\circ, 50^\circ)$ respectively. Fig.5(d) is wrapped forming a cup with flat bottom. Fig.5(e) is a pentagonal model with designed-edges. Fig.5(f) and (g) are photos of samples made by PP sheet, and Fig.5(h) is designed by elongated hexagonal hub, Fig.5(i) shows its deployed appearance.

By alternately placing clockwise and anticlockwise Archimedean spirals, inverting-type models as shown in Fig. 5(j) can be designed ($N=12$). When flexible membranes are used, it is not necessary to introduce minute Archimedean creases in these folding methods here either. By eliminating the minute Archimedean creases, a simpler foldable diagram is designed (Fig.5 (k)), where segment OCH is radial line and points B-C-D and E-F-G are on respective concentric circles. All fold lines are drawn using mirror law (see Fig.2 (d)); e.g. at point C, radial line OCH and orthogonal segment are mirror lines ($\angle ACH=\angle OCF=\phi_1$; reflection angle). At point F, the angle is ϕ_2 , where $\phi_2=\phi_1+\angle FOC$; the reflection angle ϕ becomes larger as the nodal point reaches the center O. Generally as N becomes larger, Archimedean spirals become smoother, but wrapping becomes more difficult in thick membranes. However, these difficulties can be reduced by the inverting type modeling.

3.3 Wrapping models using plane tiling and consideration of membrane sheet thickness

By replacing each circular region in Fig.5 with a regular polygon and using it as an element, we apply several kinds of regular or semi-regular plane tiling patterns in classical geometry^{10,13}. Because these elements are independently wrapped, we can design various kinds of interesting wrapping models. Fig.6 (a) and (b) are such models by only hexagons. The wrapped models look like bouquet or flowers, which are expected to be used for edutainment. This also suggests that we can design several deformed models of the famous Kawasaki-rose mathematically and systematically. Fig.6(c) shows a model of central dodecagon with surrounding hexagons and squares. These tilings are possible to expand infinitely. Another model can be designed by using tiling pattern consisting of regular dodecagons and (vacant) triangles (Fig.6(d)). We propose this model to construct several-hundred meter-sized solar sail, because the model can be manufactured in rather small building.

When the size of sail becomes large, the membrane thickness can not be ignored for wrapping. Wrapping models based on Archimedean spiral configuration have also developed by taking sheet thickness into account (Fig.6(e)). Note that the fold lines correspond to ① and ② of Fig.5(a) are not straight but are slightly curved (the curvature depends on sheet thickness). An origami model depicted in Fig.6(e) shows that circular membrane is wrapped without clearance. Based on the circular model, wrapping models for conical or parabolic shapes were designed. Examples of such models are shown in Fig.6(f) and (g), respectively¹⁴.

3.4 Design of foldable elliptical sphere and hemi sphere

As is explained in Figs.3 (l) and (m), axially foldable conical shells are designed by connecting some of the horizontal rows which satisfy the *CC* condition. By using this useful characteristic, 3-D membrane structures foldable in axial direction can easily be designed. By cutting an elliptical sphere by horizontal planes as in Fig.2 (e), round strips may be obtained, which become fan-shaped narrow strips when they are cut and opened. We denote the vertical angles of the strips by Θ_i . We further divide each strip into N -pieces of parallelogramic elements as A, B, C in Fig.2(e), satisfying the angle $\alpha = (360 - \Theta_i)/2N$. By taking out parallelogram from each strip, and stacking them up in meridian direction, an *S*-shaped part as in Fig.7 (a) is obtained (AEGF corresponds to that in Fig.3(l)). By gluing N pieces of the parts in a longitudinal direction, an axially-foldable elliptical sphere (Fig.7 (b) $N=10$) is obtained and its folded appearance is shown in Fig.7 (c).

A foldable hemi-spherical membrane was designed by the same method as the above example. Round strips are divided alternately in the inverse directions as is shown in ①~④ of Fig.2(f). By doing this, the development diagram shown in Fig.7(d) is designed ($N=12$). The zigzagged sides of the parts are designed to satisfy the folding condition in Fig.2 (f) and the V-creases in radial direction are designed to satisfy the *CC* condition. By gluing at zigzagged portions, a hemi-sphere is manufactured (Fig.7 (e)). It can be folded on the base plane, and it also shrinks in a radial direction.

4. Folding and unfolding in organism and 3-D organic Origami model

A typical wrapping model found in botany is the buds of morning glory (Fig.8 (a), (b)). Though it unfolds into a trumpet shape like the photo, design it by replacing with a conical one. Introducing a hexagonal central-hub (points A~F) with the center O (Fig.8(c)), we remove one part from hexagon. Put $\beta = 34^\circ$ and $\gamma = 86^\circ$ ($\beta + \gamma = 120^\circ$ for $N=6$) and $\alpha = 73.6^\circ$ in Fig.5 (a), and denote the points A~L as shown in Fig.8(c), where segments BM, CN, DP...and AG, BH, CI...correspond to fold lines ① and ② respectively (exchanged temporarily for M and V folding versions). Set 6 points G, H,...L so that the segments BM, CN, ...become bisectors of $\angle GBH$, $\angle HCL$... Denote 5 green regions ($\triangle ABG$, $\triangle BCH$,...) as flower ribs. By cutting off the hub part and by folding perfectly at one side of 5 ribs (BG, CH...) in the M-version and also at segments BM, CN...in the V-version, a helically stowed model for the bud of morning-glory is formed (Fig.8(d)); only rib parts appear on the surface and helical characteristic pattern in morning glory is seen. As a consequence, such helical stowing as seen in morning-glory and gentian will be modeled by the zigzagging (deformed) Archimedean spirals.

Analyses of wing folding of insects are another attractive subject by which we may study folding techniques. One of the simple but beautiful wing folding is seen in the earwig (dermaptera¹⁵) whose wing venation is shown in Fig.8 (e). Radial venations are arranged bending at a small angle

on circumferential hinge (A,...G) line. An idealized model of the radial venations with a center O is depicted in Fig.8(f), which results in equiangular spiral hinge line geometrically. Folding appearance of Fig.8 (f) is shown in Fig.8 (g). In nature, it may be seen with reference to the photograph in Fig.8 (f) that the pattern is not necessary perfectly radial; such a part (ABCD) as in Fig.4(f) may be adopted as another model. In both models, the hinge line is helical and in fact, the wing of earwig also appears to have this property.

Fig.8 (h) is a design diagram of nautilus (Fig.1 (e)) modeled by mimicking equiangular spiral in the cross section cut in half. Fig.8 (i) is designed to model the movement or deformation of flagellum, bines or tendril. These 3-D models are made by gluing top AB and bottom CD edges of the diagrams, and are shape-changeable as shown in the respective photos below. In the model (i), successive folding in the helical band (the red portion) on the cylindrical membrane bring shrink-formation then to coil. Figure 8 (j) is another model to know the growth/movement of plant tendril by using a foldable tube model of pentagonal cross section ($\alpha = \pi/5 = 36^\circ$) which is similar to Fig.3(g). When the green portion grows but the red portion is still, the tube coils. When both the green and red portions grow at the same rate, the tube is straightened. Note that the red portion shows gradual spiral. As is seen in the photo of middle-right, at the middle point of perversion, the red portion is elongated and green portion stays still as opposed to those states in coiling. This model helps the geometrical understanding of the growth/movement of plant tendril. Fig.8 (k) is a model mimicking the 3-D wound horn. To design equi-length fold lines at top and bottom edges, horizontal straight fold lines of Fig.8 (k) are deformed to equiangular spirals in this diagram. Such twisted structures with 3 or 4 regular polygonal section as this model are seen in the horns of some antelopes, e.g., the kudu (figured) or blackback, and also in some ammonites. The growth of these structures has been interpreted by the connections of gnomon elements. These models are devised to develop robotics parts of such worms as caterpillar. When these models are molded from rubber for example, and they are inflated by air pressure, they deploy a configuration. Conversely, as air pressure in the structure is decreased, they reverse the deployment described above.

5. The construction of 3-D cores combing origami with cutting

5.1 3-D honeycomb core

Honeycomb cores are common biomimetic engineering components and are widely used in aerospace structures and in architectural designs. New types of 3-D honeycomb models using only a flat sheet are presented here. Figs.9 (a)~(d) show tapered, bidirectionally tapered, aerofoil and bridge shaped core models. Fig.9 (e) is a fundamental design diagram for conventional uniformly-thick cores; solid lines are slits, while broken and dotted lines are M and V fold lines, respectively. The design may be applied by first cutting along the slits. Notice that in the vertical direction there is an alteration between M and V folds, while in the horizontal direction, the fold types alternate in pairs.

At this point the sheet should look like the photo shown next to the diagram. Glue the back of shaded square A to that of B, and also the front of A' to that of B'. Repeat the procedure for all of the other pairs along the vertical *M*-folds not separated by a slit. In doing this, a uniformly-thick core of thickness *a* is obtained. Tapered-core as Fig.9(a) is given by Fig.9(f), where the width of V-fold lines *a*, *b*, *c*, *d*...satisfy $a-b=b-c=d-c \cdots$. Consequently, vertical folds are zigzagged. A bidirectionally tapered core is designed, which is shown in Fig.9(g). For a curved surface model such as aerofoil section in Fig.9(c), a design shown in Fig.9(h), where (*a-b*), (*b-c*), (*c-d*),.....are decided by taking the change of foil curvature into account. By replacing vertical folds of Fig.9(h) into radial ones, a tapered aerofoil section is designed (Fig. 9(i)). A bridge-shaped core as Fig.9 (d) is made by the diagram shown in Fig.9(j), where wide slits result from large change of surface curvature. This is also the case for the model shown in Fig. 9(h). When these models are fixed to surface sheets, they become stable, rigid cores, but without the fixing they have foldable/deployable functions.

5.2 Sponge type 3-D core based on skew polyhedra

It is well known that only cubes within platonic polyhedra stack to completely fill space. Within semi-regular polyhedra, only truncated octahedron fills space by itself. When the use of 2 kinds of regular polyhedra are allowed, we can fill space by combining the tetrahedron and truncated tetrahedron¹⁰⁾ (Figs.10(a)~(c)). They also are called regular skew polyhedra, and often called sponge. By adapting one planar array (one column), fundamental flat cores are designed using a sheet with punchings or slittings. Figure 10(d) shows a periodically punched (black portions) sheet to make cubic type core. The sheet can be folded in two ways as in Fig.10(e) and (f). For engineering application, folding method as in Fig.10(e) is simple and useful. A cubic core made by Al alloy plate is shown in Fig.10(g). By folding a sheet with slitting as shown in Fig.10(h), an array of skew type cubic structure is obtained (Fig.10(i)) and stacking it, a cubic type skew structure is obtained. A transparent plastic made model is shown in Fig.10(j).

A core model consisting of half truncated octahedron is produced by the plane tiling consisting of regular hexagon and rhomb of apical angle 60° as shown in Fig.10(k), where every other rhomb is punched out and the remaining rhombic portions are folded into the inside of a half octahedron using horizontal slits and vertical valley folds. A half-octahedral core produced is shown in Fig.10(l). Octahedral sponge core shown can be produced by folding over a single layer model (Fig10.(l)) of planar array repeatedly until there are several layers shown as the model in Fig. 10(m).

By slitting all segments in regular hexagonal packing shown in Fig.10(n), then by folding this like origami, truncated regular tetrahedral core of skew type is produced (Fig.10(o)). Skew regular sponge structure is produced by folding over a long sheet with this pattern (Fig.10(p)).

A rather complicated 3D structure can be produced by combining folding with slitting/punching. In these examples, punched portions or slits are chosen to be as minimal as possible in order to take engineering applications into account.

6. Summary

By imitating and analyzing some organic helical patterns from geometrical and mathematical

points of view, several kinds of foldable/deployable origami models were devised. In addition, fundamental models of 3-D honeycomb cores made using a flat sheet were devised. The origami models can be applied not only to design deployable aerospace structures but to design foldable commercial products such as foldable PET bottles, cans and other engineering products. In this design process, patterns consisting of trapezoidal elements are preferred because of richer deployability and simpler creasing; e.g., Fig.3(g) is suitable for foldable plastic bottles, and they can be manufactured with current production technology. Foldable products as well as lightweight, rigid cores will promote re-use of the products and so reduce the amount of rubbish produced. To achieve this, although the author has developed a flexible metal tool to introduce numerous creases in a flat sheet by a single operation, further developments for production technology are still necessary.

The analytical origami approaches presented here may facilitate academic progress in the interpretation of plastic buckling, biomimetic robotic modelings, movable origami modelling for education or edutainment and new functional designs. Furthermore, it is expected that they will bring new interpretations of bionic mechanics. Indeed, the author envisages the formation of a new origami discipline proposing the construction of 'origami technology'. From a geometrical point of view, most patterns presented here relate to a plane tiling problem, but they are deployable to 3-D shape. Taking the geometry as a core, deployable structure, plant morphology or mechanics/growth of organism will be correlated by future studies.

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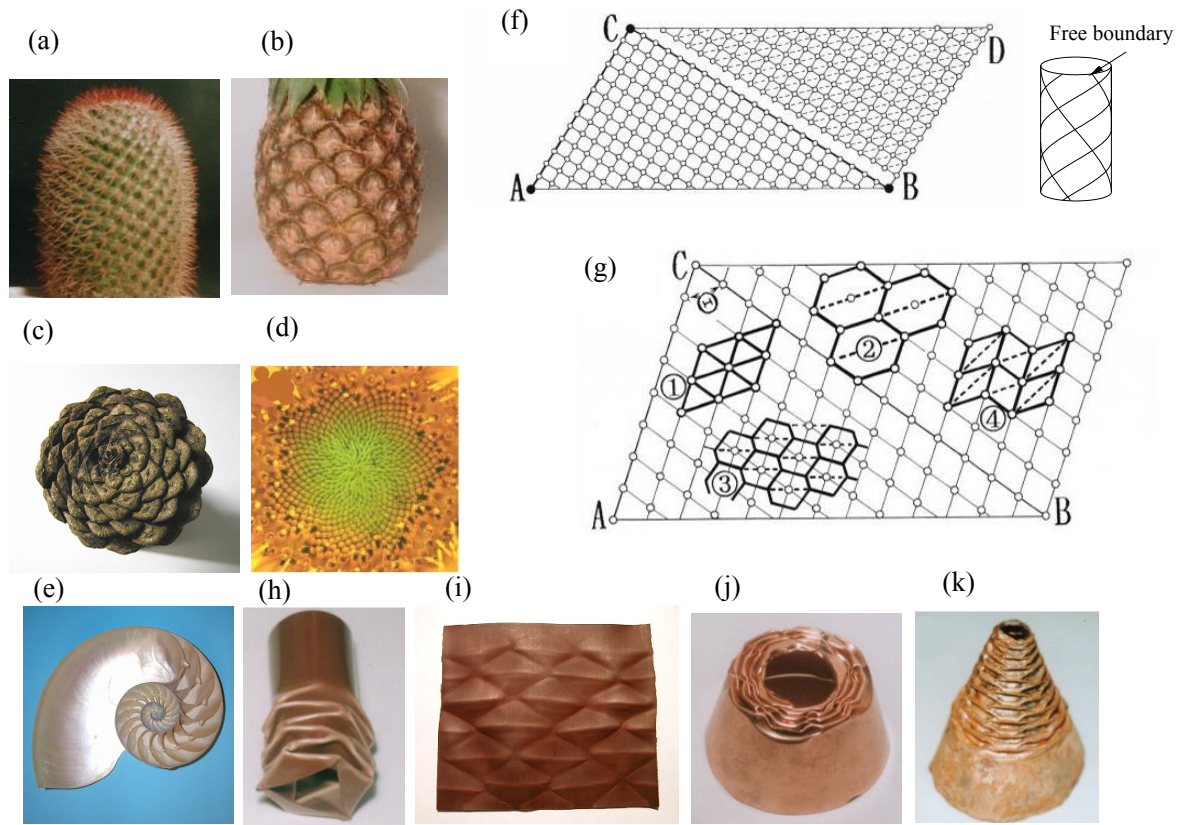


Fig.1 Some helical patterns in organisms (a)~(e), tessellations of cylindrical surface depicted by helices. Helices numbers consist of the adjoining Fibonacci number combinations (f) and (g), buckled samples of PVC tube and copper cone (h) and (j), cut/opened appearance of buckled sample (inside view) after re-stretching in boiled water using shape-memory characteristics of PVC (i), re-stretched appearance of buckled copper conical shell (k).

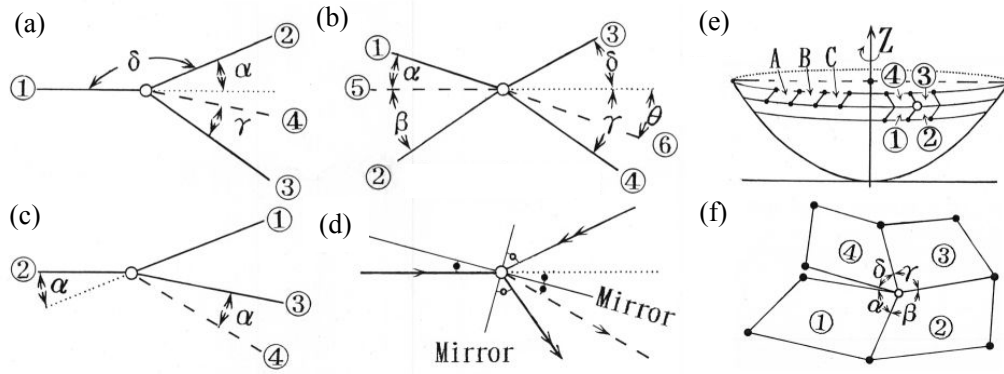


Fig.2 Folding conditions at nodes consisting of 4 and 6 fold lines (a); $\delta + \gamma = \pi$ (or $\alpha = \gamma$), (b); $\beta - \alpha = \delta - \gamma + \theta$, (c): dual-foldability (exchange of ③ and ④ of Fig.(a)), (d), orthogonal mirror law for folding (another expression of Fig.(a)), (e),(f); a model of revolutionarily shaped membranes by replacing polyhedron, and position relation of 4 elements on the plane and its folding condition at node ($\alpha + \gamma = \beta + \delta$).

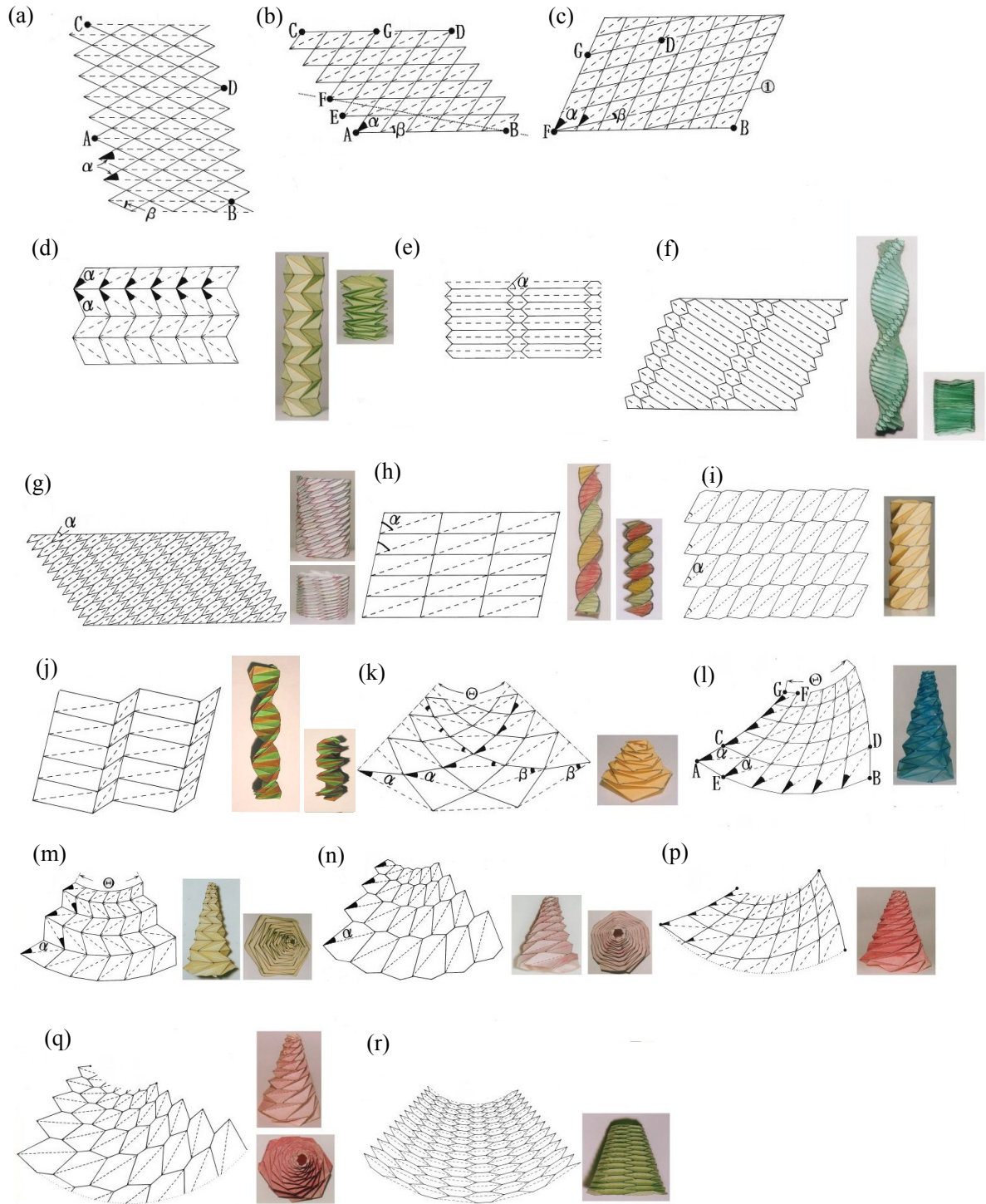


Fig.3 Examples of development diagrams for foldable/deployable tubes (a)~(j) and conical shells (k)~(q), Photos to the right of respective chart are visualizations of the corresponding folded/deployed structures. All models are perfectly foldable in the axial direction. All diagrams are helical patterns except for (a), (e), (k) and (r) which patters are named *symmetrical* (horizontal fold lines close by themselves)

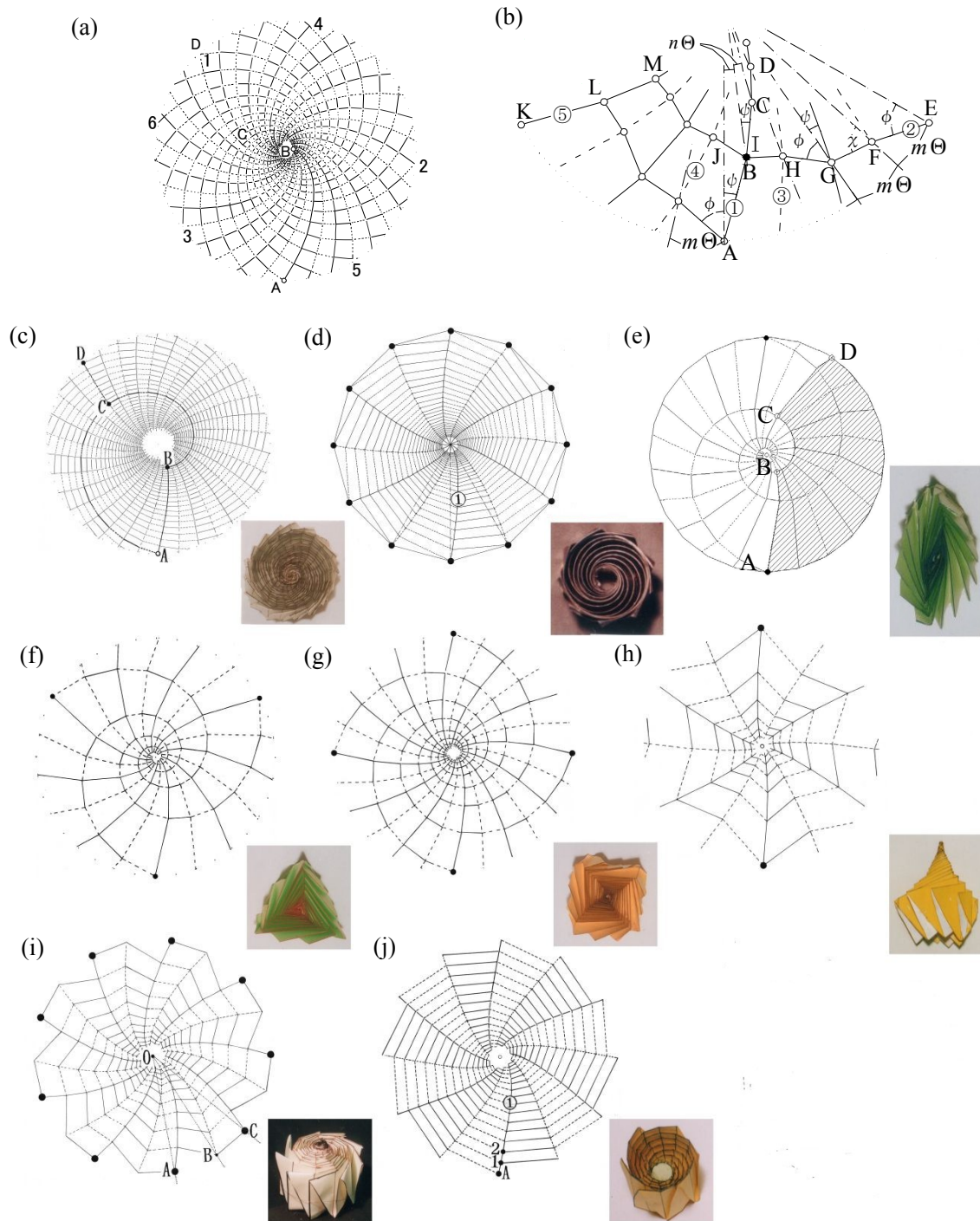


Fig.4 Wrapping models of circular membrane consisting of equiangular spirals, (a): helical model imitating sunflower (21 and 34) helices of Fig.1(d), (b); geometrical definition of angles to design foldable circular origami sample, development diagrams of foldable models and their folded samples (photos) consisting of equiangular spirals (c)~(j) or modified spirals (j)

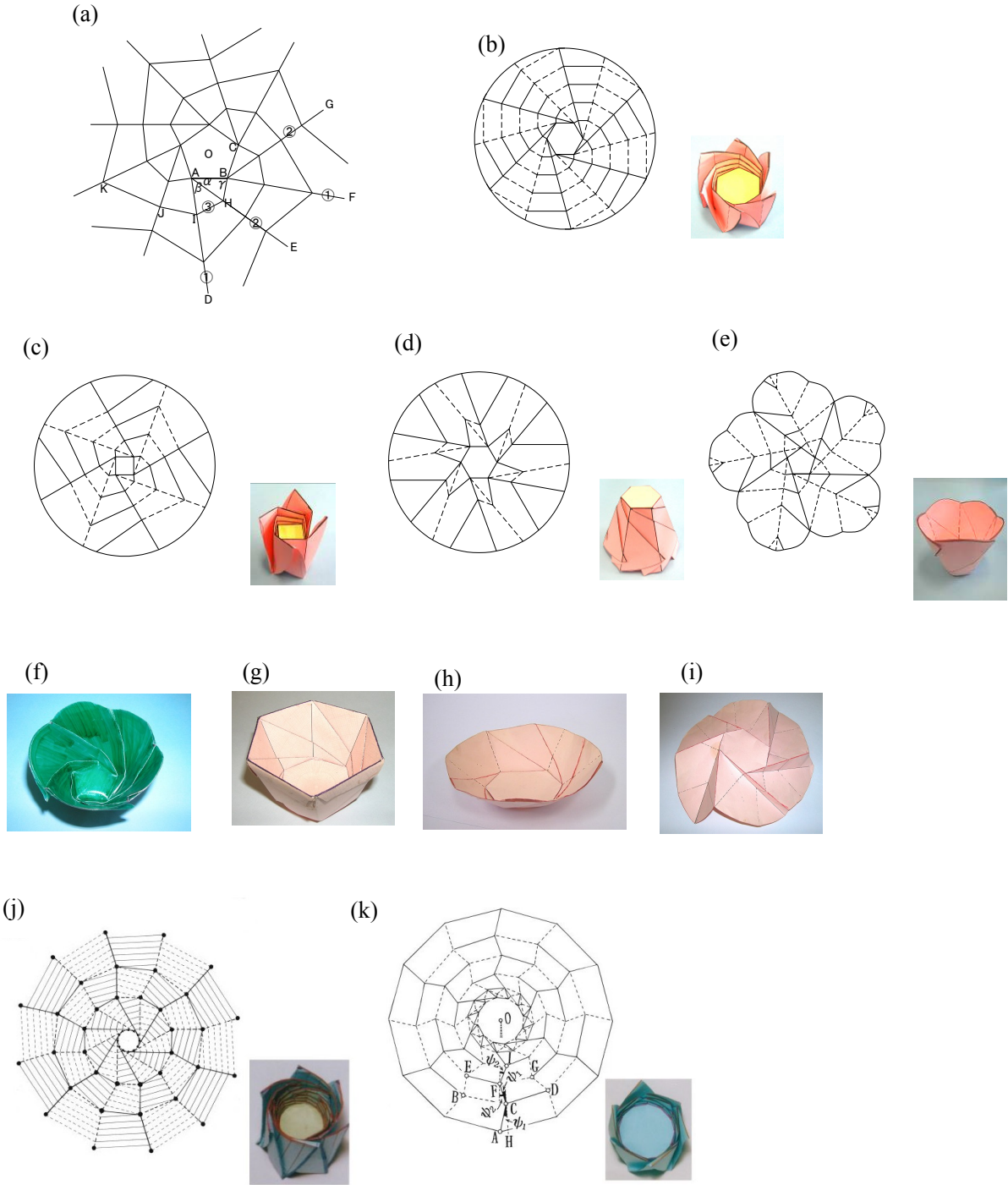


Fig.5 Wrapping models consisting of Archimedean spirals, (a) definition of angles α , β and γ to determine fold lines, ①; mountain fold, ② valley fold, (b)~(i); wrapping examples, (j),(k); inversion type folding based on Archimedean spirals configuration (for thin sheet, wrapped well without minute Archimedean spirals)

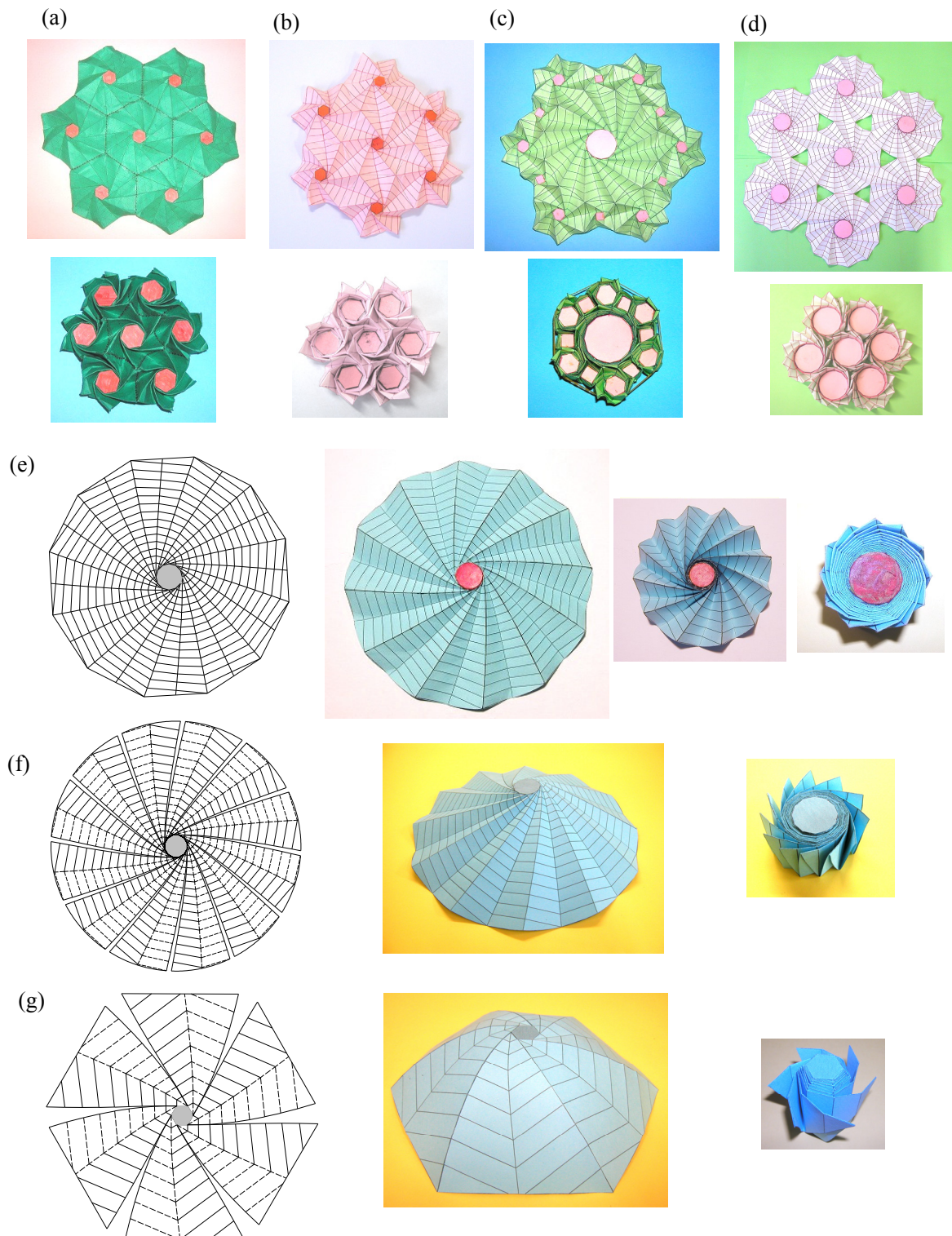


Fig.6 Wrapping models of plane tiling patterns, (a)(b); plane tiling by hexagons, (c); tiling by dodecahedron with surrounding squares and hexagons, (d); tiling by dodecagons and (vacant) triangles for constructing huge solar-sail, (e); wrapping model without clearance when wrapped (through numerical calculations with sheet thickness taken into account, fold lines ①,② slightly curved), (f)(g); conical and parabolic wrapping models taking sheet thickness into account.

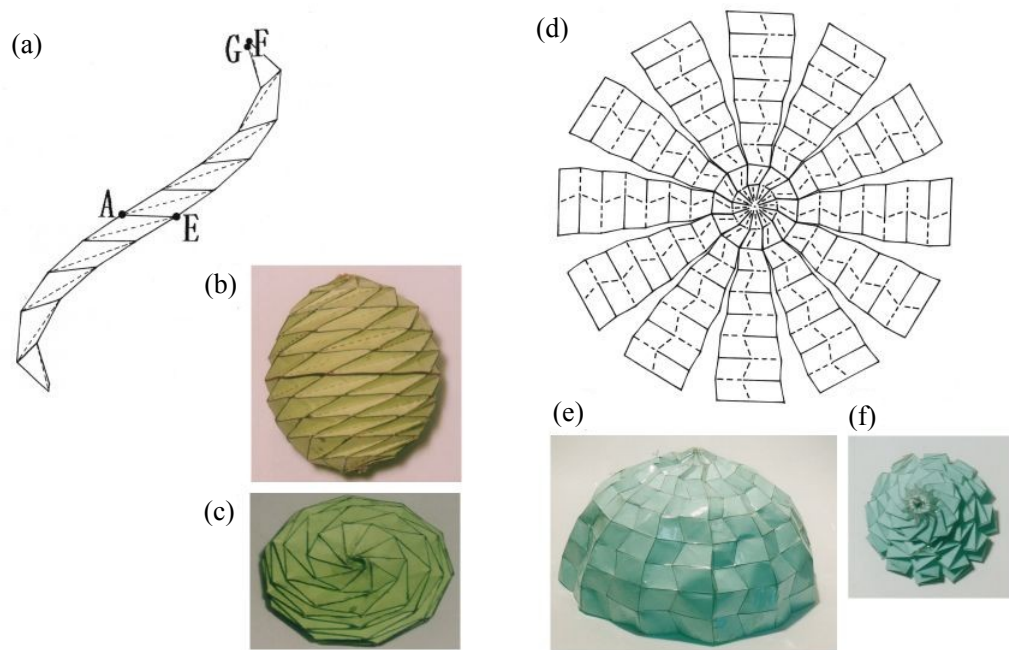


Fig.7 Foldable 3-D membrane models, (a) elliptic sphere perfectly foldable in Z-axis, designed by combining 10 parts of foldable conical shell with 5 different apical angles Θ_i , (b) axially and radially foldable hemi-sphere.

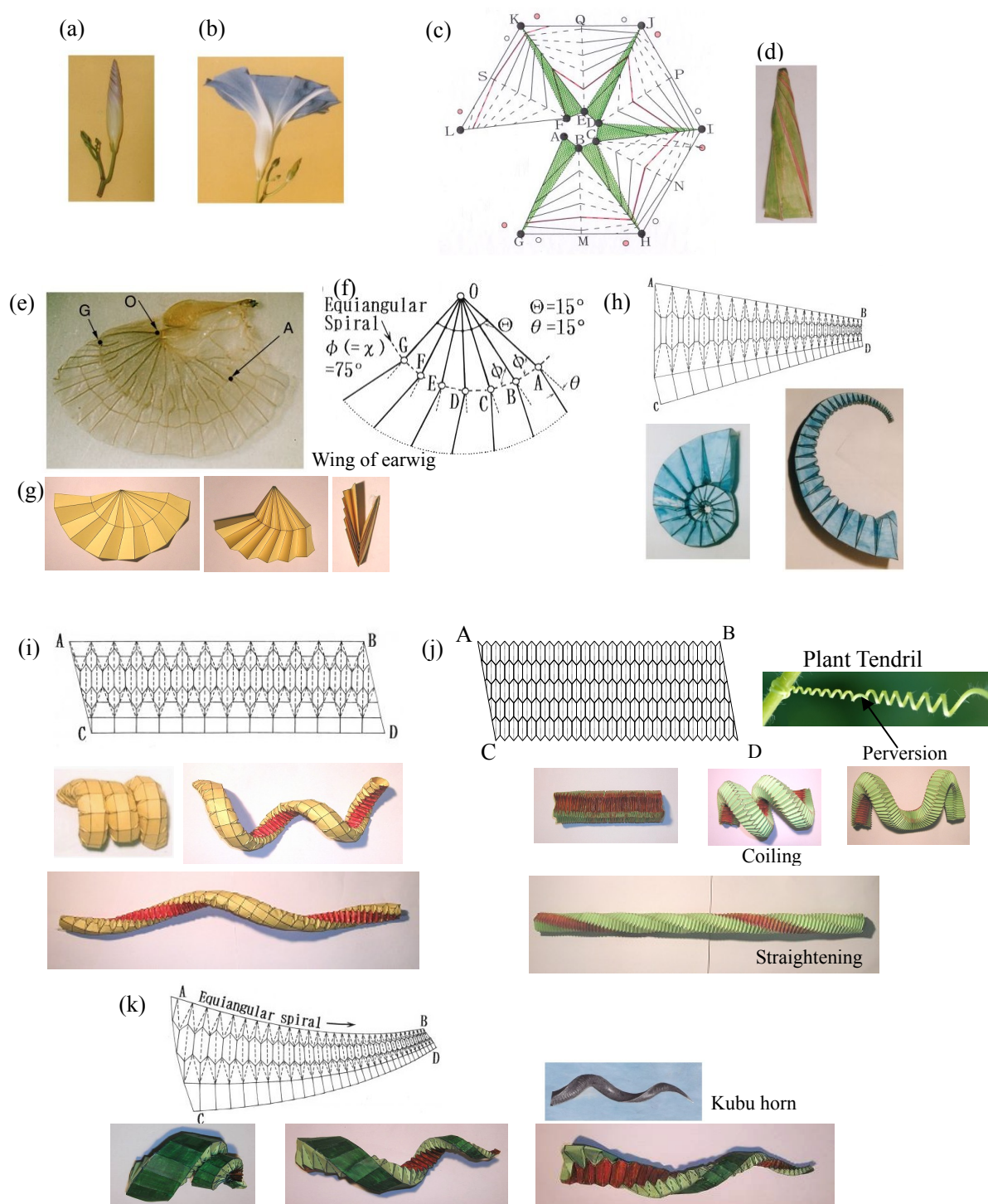


Fig.8 Examples of folding/unfolding in organisms, (a)~(d); wrapping model of the bud of morning glory by deformed Archimedean spiral configuration, (e)~(g); wing of earwig modeled by equiangular spiral configuration, (h)~(k); 3-D movable models mimicking organic structures as nautilus, flagellum, vine or tendril, and horn, respectively.

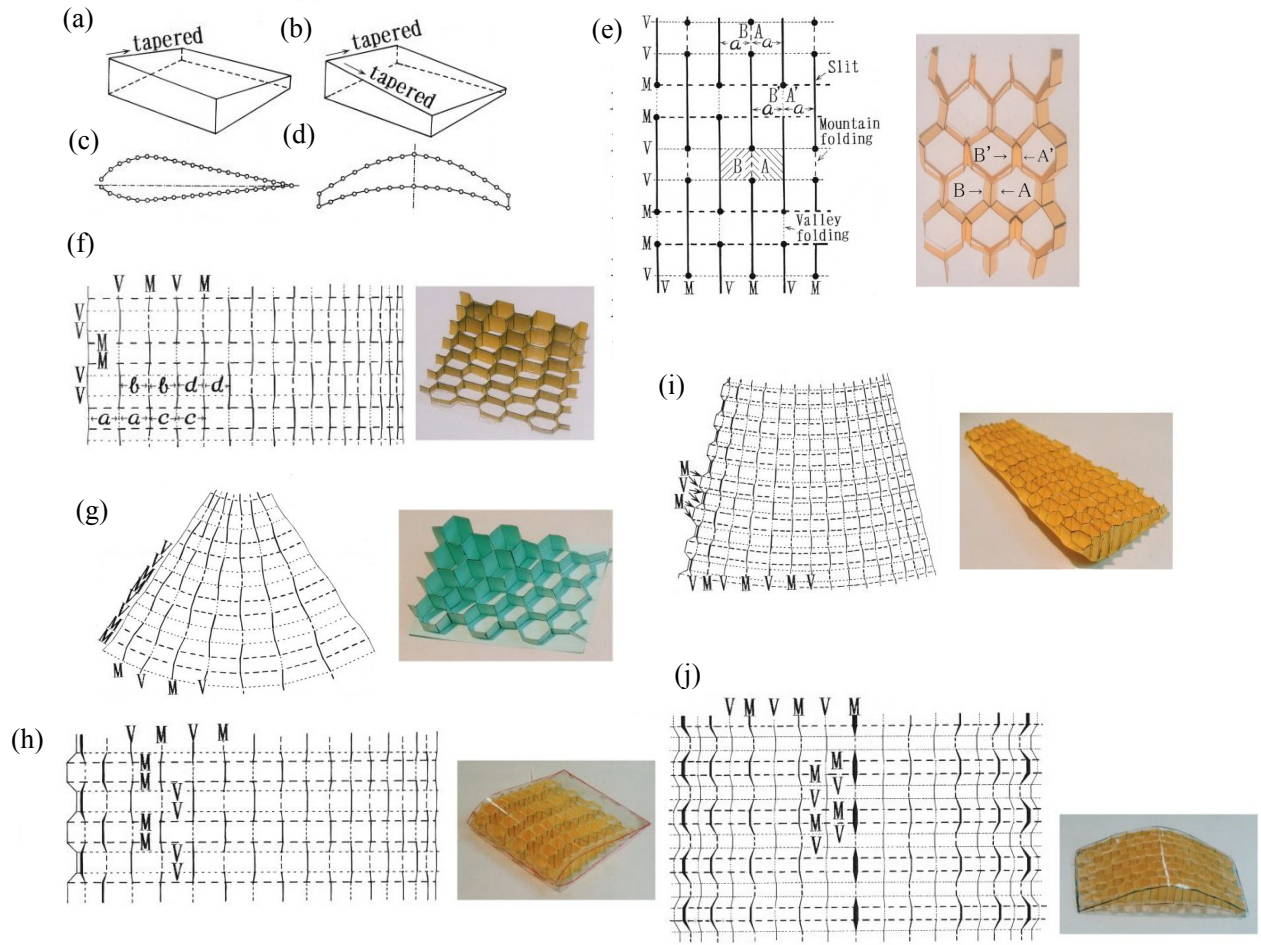


Fig.9 (a)~(d): Schematic models to design 3-D honeycomb cores, (e) fundamental diagram to make uniformly-thick core, sold lines; slits, broken lines; mountain folds (M), dotted lines; valley folds (V), (f)~(j), newly devised 3-D cores using a flat sheet foldings combining with punching or slitting (black portions or solid lines, respectively).

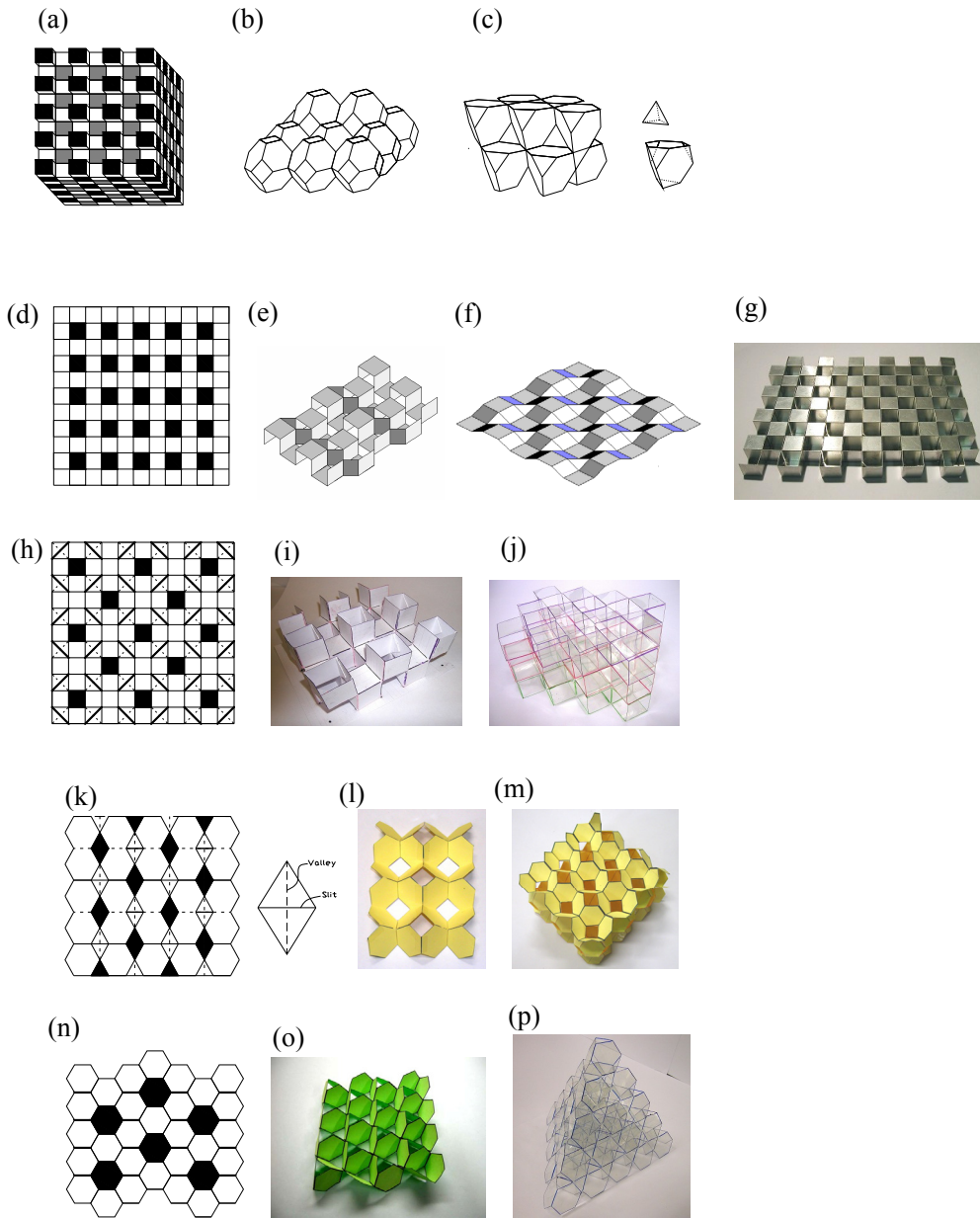


Fig.10 Sponge type cores (regular skew polyhedra), (a)~(c); schematic diagrams of sponges by cubes, truncated octahedral, and truncated tetrahedron+tetrahedron, (d) development chart to make cubic core, (e)(f); two folding methods of sheet (d), (g); manufactured new rigid, lightweight cubic core (before flange plate welded, Al alloy sheet made), (h)(i); development chart to make skew type cubic structure and its paper model, (j); plastic model of cubic sponge, (k)(l); development chart to make skew type truncated octahedral structure (black portions punched out) and its paper model, (m); truncated octahedral sponge, (n),(o) development chart to make skew type truncated tetrahedral structure and its paper model, (p); truncated tetrahedral sponge (all sponge models can be made by a single sheet)